10. For what values of 'a' the game with the following pay-off matrix is strictly determinable ?
[JNTU (Mech. & Prod.) 2004]

$$\begin{array}{cc|ccc|} & & & B & \\ & & B_1 & B_2 & B_3 \\ \hline & A_1 & a & 6 & 2 \\ A & A_2 & -1 & a & -7 \\ & A_3 & -2 & 4 & a \\ \hline \end{array}$$

19.8. RECTANGULAR GAMES WITHOUT SADDLE POINT

As discussed earlier, if the payoff matrix $\{v_{ij}\}$ has a saddle point (r, s), then $i = r, j = s$ are the optimal strategies of the game and the payoff v_{rs} $(= v)$ is the value of the game. On the other hand, if the given matrix has no saddle point, the game has no optimal strategies. The concept of optimal strategies can be extended to all matrix games by introducing a probability with choice and mathematical expectation with payoff.

Let player A choose a particular activity i such that $1 \le i \le m$ with probability x_i. This can also be interpreted as the relative frequency with which A chooses activity i from number of activities of the game. Then set $\mathbf{x} = \{x_i, 1 \le i \le m\}$ of probabilities constitute the strategy of A. Similarly, $\mathbf{y} = \{y_j, 1 \le j \le n\}$ defines the strategy of the player B.

Thus, the vector $\mathbf{x} = (x_1, x_2, \dots, x_m)$ of non-negative numbers satisfying $x_1 + x_2 + \dots + x_m = 1$ is called the *mixed strategy of the player A*. Similarly, the vector $\mathbf{y} = (y_1, y_2, \dots, y_n)$ of non-negative numbers satisfying $y_1 + y_2 + \dots + y_n = 1$ is called the *mixed strategy of the player B.*

Consider the symbol S_m which denotes the set of ordered m-tuples of non-negative numbers whose sum is unity and $\mathbf{x} \in S_m$. Similarly, $\mathbf{y} \in S_n$. Unless otherwise stated, assume that $\mathbf{x} \in S_m$ and $\mathbf{y} \in S_n$, where $\mathbf{x}$ and $\mathbf{y}$ are mixed strategies of player A and B, respectively.

The mathematical expectation of the payoff function $E(\mathbf{x}, \mathbf{y})$ in a game whose payoff matrix is $\{v_{ij}\}$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i\, v_{ij})\, y_j \qquad = \mathbf{x}^T \mathbf{v}\, \mathbf{y} \quad (\textit{in matrix form})$$

where **x** *and* **y** *are the mixed strategies of players A and B*, respectively,

Thus the player A should choose **x** so as to maximize his minimum expectation and the player B should choose **y** so as to minimize the player A's greatest expectation. In other words, the player

A tries for $\max\limits_{\mathbf{x}} \min\limits_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$ and B tries for $\min\limits_{\mathbf{y}} \max\limits_{\mathbf{x}} E(\mathbf{x}, \mathbf{y})$.

At this stage it is possible to define the *strategic saddle point* of the game with mixed strategies.

Strategic Saddle Point. ***Definition.*** *If* $\min\limits_{\mathbf{y}} \max\limits_{\mathbf{x}} E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}_0, \mathbf{y}_0) = \max\limits_{\mathbf{x}} \min\limits_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$, *then* $(\mathbf{x}_0, \mathbf{y}_0)$ *is called the* ***strategic saddle point*** *of the game where* $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *define the optimal strategies, and* $v = E(\mathbf{x}_0, \mathbf{y}_0)$ *is the value of the game.*

According to the *minimax theorem* (***Section 19·11***), a strategic saddle point will always exist.

Example 7. *In a game of matching coins with two players, suppose one player wins Rs. 2 when there are two heads and wins nothing when there are two tails; and losses Re. 1 when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.*

Solution. The payoff matrix (for the player A) is given by

$$\begin{array}{ccc|cc|c} & & & \text{Player } B & & \\ & & & H & T & \text{Row Min.} \\ \hline & & H & 2 & -1 & -1 \\ \text{Player } A & & & & & \\ & & T & -1 & 0 & -1 \\ \hline & \text{Column Max :} & & 2 & 0 & \end{array}$$

$\rangle$ Maximin (lower) value ($\underline{v}$)

$\uparrow$ Minimax (upper) value ($\overline{v}$)

Here, maximin value $(\underline{v}) = -1 \neq$ minimax value $(\overline{v}) = 2$.

So the matrix is whithout saddle point.

Now, let us outline here how one finds the best strategies for such games and the expected amounts to be gained or lost by the players.

Let the player A plays H with probability x and T with probability $1 - x$ so that $x + (1 - x) = 1$. Then, if the player B plays H all the time, A's expected gain will be

$$E(A, H) = x.2 + (1 - x)(-1) = 3x - 1. \qquad \dots(19{\cdot}6)$$

Similarly, if the player B plays T all the time, A's expected gain will be

$$E(A, T) = x(-1) + (1-x)\,0 = -x. \qquad \text{...(19·7)}$$

It can be shown mathematically that if the player A chooses x such that

$$E(A, H) = E(A, T) = E(A), \text{ say}, \qquad \text{...(19·8)}$$

then this will determine the best strategy for him.

Thus, $$3x - 1 = -x \text{ or } x = 1/4 \qquad \text{...(19·9)}$$

Therefore, best strategy for the player A is to play H and T with probability 1/4 and 3/4, respectively. Since this is a mixed strategy, it is usually denoted by the set {1/4, 3/4}. So expected gain for the player A is given by

$$E(A) = \frac{1}{4}\cdot 2 + \frac{3}{4}(-1) = -\frac{1}{4}$$

Now, whatever be the set $\{y, 1-y\}$ of probabilities with which the player B plays either H or T, A's expected gain will always remain equal to $-1/4$. To verify this,

$$E(A, y, 1-y) = y\left[\frac{1}{4}\cdot 2 + \frac{3}{4}(-1)\right] + (1-y)\left[\frac{1}{4}(-1) + \frac{3}{4}\,0\right]$$

$$= y\left(-\frac{1}{4}\right) + (1-y)\left(-\frac{1}{4}\right) = -\frac{1}{4}. \qquad \text{...(19·10)}$$

The same procedure can be applied for the player B. Let the probability of the choice of H be denoted by y and that of T by $(1-y)$. For best strategy of the player B,

$$E(B, H) = E(B, T) = E(B), \text{ say} \qquad \text{...(19·11)}$$

or $$y \cdot 2 + (1-y)(-1) = y(-1) + (1-y)\,0$$

or $$4y = 1$$

or $$y = 1/4 \quad \text{and} \quad \text{therefore} \quad 1 - y = 3/4.$$

Therfore, $E(B) = \frac{1}{4}\, 2 + \frac{3}{4}(-1) = -\frac{1}{4}$.

Here, $E(A) = E(B) = -1/4$. Thus, the complete solution of the game is :

(*i*) The player A should play H and T with probabilities 1/4 and 3/4, respectively. Thus, A's optimal strategy is $\mathbf{x}_0 = (1/4, 3/4)$.

(*ii*) The player B should play H and T with probabilities 1/4 and 3/4; respectively. Thus, B's optimal strategy is $\mathbf{y}_0 = (1/4, 3/4)$.

(*iii*) The expected value of the game is $-1/4$ to the player A. Here $(\mathbf{x}_0, \mathbf{y}_0)$ is the strategic saddle point of this game.

Remark. Although this example can be easily solved by using the formula of **Section** 19·13, the present discussion will be of great help in understanding the further discussion.

EXAMINAITON PROBLEMS

1. Find the optimal strategies for the games for which the pay off matrices are given below. Also, find the value of the game.

(a)
$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{P_2} \\ & & \text{I} & \text{II} \\ \hline P_1 & \text{I} & 1 & 3 \\ & \text{II} & 4 & 2 \end{array}$$

[**Ans.** (1/2 , 1/2) , (1/4 , 3/4) ; $v = 5/2$]

(b)
$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{P_2} \\ & & \text{I} & \text{II} \\ \hline P_1 & \text{I} & -4 & 6 \\ & \text{II} & 2 & -3 \end{array}$$

[**Ans.** (1/3 , 2/3) , (3/5 , 2/5); $v = 0$]

2. For the game with the following payoff matrix for the row player, determine the optimal strategies for both the players and the value of the game :

(a) $$\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}$$

[**Ans.** (1/4 , 3/4) , (1/4 , 3/4) ; $v = -3/4$]

(b) $$\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$$

[**Ans.** (2/5 , 3/5) , (1/2 , 1/2) $v = 4$]

3. A game has the payoff matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Show that $E(x, y) = 1 - 2x\left(y - \frac{1}{2}\right)$ and deduce that in the solution of the game the first player follows a pure strategy while the second has infinite number of mixed strategies. **[Raj. (M.Phil.) 92]**

4. State the fundamental theorem of rectangular games. Show that $\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$ in the arbitrary matrix :

$$\begin{pmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots \qquad \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{pmatrix}$$

19.9. MINIMAX-MAXIMIN PRINCIPLE FOR MIXED STRATEGY GAMES

It has been observed earlier that if a game does not have a saddle point, two players cannot use the maximin-minimax (pure) strategies as their optimal strategies. This failure of the minimax-maximum (pure) strategies, in general, give an optimal solution to the game and led to the idea of using mixed strategies. Each player, instead of selecting pure strategies only, may play all his strategies according to a predetermined set of probabilities.

Let $x_1, x_2, \dots, x_m$ and $y_1, y_2, y_3, \dots y_n$ be the probabilities of two players A and B, respectively to select their pure strategies.

Then, $$x_1 + x_2 + x_3 + \dots + x_m = 1 \qquad \text{...(19·12)}$$

and $$y_1 + y_2 + y_3 + \dots + y_n = 1, \qquad \text{...(19·13)}$$

where $x_i \geq 0$ and $y_j \geq 0$ for all i and j. Thus if v_{ij} represents the (i, j)th entry of the game matrix, probabilities x_i and y_j will appear (*Table 19·7*).

Table 19·7

Player B

	Probabilties→ ↓	j \ i	y_1	y_2	...	y_j	...	y_n
			1	2	...	j	...	n
	x_1	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	x_2	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player A	:	:	:	:		:		:
	x_i	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:	:		:		:
	x_m	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

The solution of mixed strategy problem is also based on the minimax criterion given in ***Section 19·4.*** The only difference is that the player A selects probabilities x_i which *maximize his minimum 'expected' gain* in a column, while the player B selects the probabilities which *minimize his maximum 'expected' loss* in a row.

Mathematically, the minimax criterion for a mixed strategy is as follows :

The player A selects x_i $(x_i \geq 0, \sum_{i=1}^{m} x_i = 1)$ which gives the lower value of the game

$$\max_{x_1, x_2, \dots, x_m} [\min \{(v_{11}x_1 + v_{21}x_2 + \dots + v_{m1}x_m), (v_{12}x_1 + v_{22}x_2 + \dots + v_{m2}x_m), \dots, (v_{1n}x_1 + v_{2n}x_2 + \dots + v_{mn}x_m)\}] \qquad \text{...(19·14a)}$$

or more precisely,

$$\underline{v} = \max_{x_i} \left[\min \left\{ \sum_{i=1}^{m} v_{i1}\, x_i\,,\ \sum_{i=1}^{m} v_{i2} x_i\,, \dots,\ \sum_{i=1}^{m} v_{in}\, x_i \right\} \right] \qquad \text{...(19·14b)}$$

Similarly, the player B chooses y_j $(y_j \geq 0, \sum_{j=1}^{n} y_j = 1)$ which gives the upper value of the game

$$\overline{v} = \min_{y_j} \left[\max \left\{ \sum_{j=1}^{n} v_{1j}\, y_j\,,\ \sum_{j=1}^{n} v_{2j}\, y_j\,, \dots,\ \sum_{j=1}^{n} v_{mj}\, y_j \right\} \right] \qquad \text{...(19·15)}$$

These values are referred to the *maximin* ($\underline{v}$) and the *minimax* ($\overline{v}$) expected values, respectively.

In pure strategies, the relationship, $\overline{v} \geq \underline{v}$, holds in general. When x_i and y_j correspond to the optimal solution, this relation holds in 'equality' sense and the 'expected' values thus obtained become equal to the (optimal) expected values of the game. This result follows from the *minimax theorem* (called the *fundamental theorem of rectangular games*) which is derived in ***Sec. 19·12.***

We shall require the following *Lemma* in ***Sec 19·10.***

Lemma. *Let* A $= (v_{ij})$ *be the payoff matrix of an* $m \times n$ *game. If* B $= (v'_{ij})$ *is obtained from A by adding a constant c to every element of A, then an optimal strategy for B is also an optimal strategy for A.*

Proof. Let v' be the value of the game with payoff matrix **B**. Then for the strategies **x, y,**

$$\sum_{i=1}^{m} \sum_{j=1}^{n} v'_{ij} x_i y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} x_i y_j + c .$$

If **x*** , **y*** are optimal strategies for the game **B**, then

$$\sum_i \sum_j v'_{ij} x_i y_j^* \le v' \le \sum_i \sum_j v'_{ij} x_i^* y_j \Rightarrow \sum_i \sum_j v_{ij} x_i y_j^* + c \le v' \le \sum_i \sum_j v_{ij} \; x_i^* y_j + c$$

$$\therefore \qquad \sum_i \sum_j v_{ij} x_i y_j^* \le v' - c \le \sum_i \sum_j v_{ij} x_i^* y_j .$$

Thus $\mathbf{x}^*$, $\mathbf{y}^*$ are optimal for game **A** with the value of game $v = v' - c$.

Hence arbitrarily chosen constant c can be added to each element of **A** and then we can solve the resulting game **B**. The value v of the original game is then obtained simply by subtracting the constant c from the value of the game **B**. Constant c is chosen so large that $v_{ij} + c$ is positive (> 0) for all i and j, so that the value of the game is certainly positive.

Q. 1. Define the terms "maximin element, minimax element and saddle point" of the payoff matrix of a two-person zero-sum games.

2. Explain 'minimax criterion' as applied to the theory of games. **[Bhubneshwar (IT) 2004]**

3. Let (v_{ij}) be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then prove that $\overline{v} \ge \underline{v}$. That is, $\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}]$.

[Meerut (Stat.) 90]

EXAMINATION PROBLEMS

Find the minimax and maximin value of the following games :

(i) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 7 & -1 & 3 \\ 4 & 8 & 0 & -6 \\ 6 & -9 & -2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & 9 & 6 & 8 \\ -2 & 10 & 4 & 6 \\ 5 & 3 & 0 & 7 \\ 7 & -2 & 8 & 4 \end{bmatrix}$

[Madurai B.Sc. (Comp. Sc.) 92]

[Ans. (i) minimax = 3, maximin = 1, (ii) minimax = 0, maximin = – 1, (iii) $2 < v < 4$, (iv) $4 \le v < 7$]

19.10. EQUIVALENCE OF RECTANGULAR GAME AND LINEAR PROGRAMMING

It has been shown that the player A chooses his optimum mixed stratgies in order to *maximize his minimum 'expected' gain, i.e.*

$$\max_{x_i} \left[\min \left\{ \sum_{i=1}^{m} v_{i1} x_i , \sum_{i=1}^{m} v_{i2} x_i , \ldots , \sum_{i=1}^{m} v_{in} x_i \right\} \right] \qquad \ldots(19 \cdot 16)$$

subject to the constraints :

$$x_1 + x_2 + x_3 + \ldots + x_m = 1 \qquad \ldots(19 \cdot 17)$$

$$x_i \ge 0, i = 1, 2, \ldots , m . \qquad \ldots(19 \cdot 18)$$

Now, in order to express this problem in linear programming form, let

$$\min \left[\sum_{i=1}^{m} v_{i1} x_i , \sum_{i=1}^{m} v_{i2} x_i , \ldots , \sum_{i=1}^{m} v_{in} x_i \right] = v \qquad \ldots(19 \cdot 19)$$

which immediately implies that

$$\sum_{i=1}^{m} v_{i1} x_i \ge v , \sum_{i=1}^{m} v_{i2} x_i \ge v , \ldots , \sum_{i=1}^{m} v_{in} x_i \ge v . \qquad \ldots(19 \cdot 20)$$

Thus, the problem now becomes :Maximize $x_0 = v$ subject to the constraints :

$$\left.\begin{array}{rr} & v_{11}\,x_1+v_{21}\,x_2+v_{31}\,x_3+\ldots+v_{m1}\,x_m \geq v \\ & v_{12}\,x_1+v_{22}\,x_2+v_{32}\,x_3+\ldots+x_{m2}\,x_m \geq v \\ & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & v_{1n}\,x_1+v_{2n}\,x_2+v_{3n}\,x_3+\ldots+v_{mn}\,x_m \geq v \\ & x_1+x_2+x_3+\ldots+x_m=1 \\ \text{and} & x_1\,,x_2\,,x_3\,,\ldots,x_m \geq 0 \end{array}\right\} \qquad \ldots(19{\cdot}21a)$$

Here v represents the value of the game. This linear programming formulation can be simplified by dividing all $(n+1)$ constraints by v ; **the division** is valid as long as $v > 0$* . In case, $v < 0$, the direction of the inequality constraints must be reversed, **and** if $v=0$, division would be meaningless. The later point creates no special difficulty since a constant c **can be added** to all entries of the matrix ensuring that the value (v) of the game for the 'revised' matrix becomes **greater than** zero. After the optimal solution is obtained, the true value of the game is obtained by subtracting **the same amount** c.

In general, if the maximum value of the game is non-negative, the value of the game is greater than zero (provided the game does not have a saddle point). Thus, assuming $v > 0$, the constraints become :

$$\left.\begin{array}{r} v_{11}\dfrac{x_1}{v}+v_{21}\dfrac{x_2}{v}+\ldots+v_{m1}\dfrac{x_m}{v} \geq 1 \\ v_{12}\dfrac{x_1}{v}+v_{22}\dfrac{x_2}{v}+\ldots+v_{m2}\dfrac{x_m}{v} \geq 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n}\dfrac{x_1}{v}+v_{2n}\dfrac{x_2}{v}+,\ldots+v_{mn}\dfrac{x_m}{v} \geq 1 \\ \dfrac{x_1}{v}+\dfrac{x_2}{v}+\ldots+\dfrac{x_m}{v}=\dfrac{1}{v} \end{array}\right\} \qquad \ldots(19{\cdot}21b)$$

Now, suppose $\frac{x_1}{v}=X_1\,,\frac{x_2}{v}=X_2\,,\ldots,\frac{x_m}{v}=X_m$, and $\frac{1}{v}=\mathbf{x_0}$, then

$$\max v=\min\left(\frac{1}{v}\right)=\min\left\{\frac{x_1}{v}+\frac{x_2}{v}+\ldots+\frac{x_m}{v}\right\} \qquad \ldots(19{\cdot}22)$$

$= \min\,\{X_1+X_2+X_3+\ldots+X_m\}$(which is justified by the last constraint),

Now, finally, the equivalent LP problem becomes :

Minimize $\mathbf{x_0}=X_1+X_2+\ldots+X_m$, subject to the constraints : ...(19·23)

$$\left.\begin{array}{r} v_{11}\,X_1+v_{21}\,X_2+\ldots+v_{m1}\,X_m \geq 1 \\ v_{12}\,X_1+v_{22}\,X_2+\ldots+v_{m2}\,X_m \geq 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n}\,X_1+v_{2n}\,X_2+\ldots+v_{mn}\,X_m \geq 1 \\ X_1 \geq 0\,,X_2 \geq 0\,,\ldots,X_m \geq 0 \end{array}\right\} \qquad \ldots(19{\cdot}21c)$$

After an optimal solution is obtained by the simplex method, original optimal values can be obtained from the given transformation formulae.

On the other hand, player B chooses his mixed strategies in order to *minimize his maximum 'expected' loss, i.e.*

$$\min_{y_j}\left[\max\left\{\sum_{j=1}^{n} v_{1j}\,y_j\,,\sum_{j=1}^{n} v_{2j}\,y_j\,,\ldots,\sum_{j=1}^{n} v_{mj}\,y_j\right\}\right] \qquad \ldots(19{\cdot}24)$$

subject to the constraints :

$$y_1+y_2+\ldots+y_n=1 \qquad \ldots(19{\cdot}25)$$

$$y_1 \geq 0\,,y_2 \geq 0\,,\ldots,y_n \geq 0. \qquad \ldots(19{\cdot}26)$$

---

* For convenience, in order to convert a matrix game into a linear programming problem, first make all entries of the matrix positive by adding a positive constant c to all elements of the matrix game. Of course, c will be subtracted later on from the value of the game v.

Proceeding in the like manner, linear programming form of the B's problem becomes :

Maximize $\mathbf{y}_0 = Y_1 + Y_2 + \ldots + Y_n$, subject to the constraints : ...(19·27)

$$\left.\begin{array}{l} v_{11}\,Y_1 + v_{12}\,Y_2 + \ldots + v_{1n}\,Y_n \le 1 \\ v_{21}\,Y_1 + v_{22}\,Y_2 + \ldots + v_{2n}\,Y_n \le 1 \\ \;\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{m1}\,Y_1 + v_{m2}\,Y_2 + \ldots + v_{mn}\,Y_n \le 1 \\ Y_1 \ge 0\,, Y_2 \ge 0\,, \ldots, Y_n \ge 0 \end{array}\right\} \quad \ldots(19{\cdot}28)$$

where $$\mathbf{y}_0 = \frac{1}{v}\,,\; Y_1 = \frac{y_1}{v}\,, Y_2 = \frac{y_2}{v}\,, \ldots, Y_n = \frac{y_n}{v}\,.$$

Further, it has been observed that the player B's problem is exactly the dual of the player A's problem. The optimal solution of one problem will automatically give the optimal solution to the other and that min $\mathbf{x}_0$ = max $\mathbf{y}_0$. The player B's problem can be solved by regular simplex method while player A's problem can be solved by the dual simplex method.

The choice of either method will depend on which problem has a smaller number of constraints. This in turn depends on the number of pure strategies for either player.

Q. 1. Show how a 'game' can be formulated as a linear programming problem. **[IAS (Maths.) 99; Raj. Univ. (M. Phil) 90]**

2. With the help of an appropriate example establish the relationship between 'Game theory' and 'Linear Programming'.

3. Establish the relation between a linear programming problem and a two-person zero–sum game. **[Meerut (OR) 2003]**

4. Discuss equivalence of matrix game and the problem of linear programming. **[Kanpur M.Sc. (Math.) 97; Delhi (OR.) 95; Banasthali (M.Sc.) 93]**

5. Explain the method of solving a zero-sum two person game as a linear programming problem. **[Meerut 2005; Delhi 90]**

6. Establish the equivalence of matrix game and the problem of linear programming. **[Delhi B.Sc. (Math) 93]**

19.11. MINIMAX THEOREM (FUNDAMENTAL THEOREM OF GAME THEORY)

Theorem 19·2. ***(Fundamental Theorem of Rectangular Games).*** *If mixed strategies are allowed, there always exists a value of the game, i.e.* $\overline{v} = \underline{v} = v$.

Alternative Statement. *If* $\Sigma x_i = \Sigma y_j = 1,\ x_i \ge 0,\ y_j \ge 0$, *then*

$$\max_{\mathbf{x}}\ \min_{\mathbf{y}\,|\,\mathbf{x}}\ \sum_i \sum_j v_{ij}\,(x_i\,y_j) = \min_{\mathbf{y}}\ \max_{\mathbf{x}\,|\,\mathbf{y}}\ \sum_i \sum_j v_{ij}\,(x_i\,y_j)\,,$$

where the symbol **y | x** *means "***y** *given* **x***". The left side relates that for some fixed (given)* **x***, minimize the sum with respect to* **y***. This results in a value showing it is a function of* **x***, select* **x** *so that this value is maximum.*

Proof. The player A's problem (from **sec. 19·10**) is :

$$\text{Min. } \mathbf{x}_0 = X_1 + X_2 + X_3 + \ldots + X_m \text{, subject to}$$

$$\begin{array}{l} v_{11}X_1 + v_{21}X_2 + \ldots + v_{m1}X_m \ge 1 \\ v_{12}X_1 + v_{22}X_2 + \ldots + v_{m2}X_m \ge 1 \\ \;\vdots \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ v_{1n}X_1 + v_{2n}X_2 + \ldots + v_{mn}X_m \ge 1 \\ X_1 \ge 0, X_2 \ge 0, \ldots, X_m \ge 0\,. \end{array}$$

The dual problem corresponding to above linear programming problem (called the primal problem) is :

$$\text{Max. } \mathbf{y}_0 = Y_1 + Y_2 + Y_3 + \ldots + Y_n \text{, subject to}$$

$$\begin{array}{l} v_{11}Y_1 + v_{12}Y_2 + \ldots + v_{1n}Y_n \le 1 \\ v_{21}Y_1 + v_{22}Y_2 + \ldots + v_{2n}Y_n \le 1 \\ \;\vdots \qquad \vdots \qquad\quad \vdots \quad \vdots \\ v_{m1}Y_1 + v_{m2}Y_2 + \ldots + v_{mn}Y_n \le 1 \\ Y_1 \ge 0\,, Y_2 \ge 0\,, \ldots, Y_n \ge 0\,. \end{array}$$

It has been seen that this dual problem is similar to the problem obtained for the player *B* in *Sec.* ***19·10.***

But, the duality theorem states that :

If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution, and optimum numerical values of the objective function are equal, i.e.

$$\max \mathbf{y_0} = \min \mathbf{x_0} \quad \text{or} \quad \underline{v} = \overline{v} = v \text{ (value of the game)}$$

This completes the proof of the theorem.

Q. 1. State, explain and prove the 'minimax theorem' (fundamental theorem) for two-person zero-sum finite games.

[Kanpur M.Sc (Math.) 96; Delhi (OR) 93]

2. Let v be the value of a rectangular game with payoff matrix $B = (p_{ij})$. Show that $\min p_{ji} \le v \le \max (p_{ij})$ and $\max_i \min_j p_{ij} \le v \le \min_j \max_i p_{ji}$.

3. Let E (**p**, **q**) be expectation function in an m × n matrix rectangular game between player A and B, such that $\mathbf{p} \in R^n$, $\mathbf{q} \in R^m$. If E (**p, q**) be such that both $\max_{\mathbf{p}} \min_{\mathbf{q}}$ E (**p, q**) and $\min_{\mathbf{q}} \max_{\mathbf{p}}$ E (**p, q**) exist, then show that

$\min_{\mathbf{q}} \max_{\mathbf{p}}$ E (**p, q**) ≥ $\max_{\mathbf{p}} \min_{\mathbf{q}}$ E (**p, q**) (**p** and **q** are probability vectors)

[Raj. Univ. (M. Phil.) 91]

19·12 SOLUTION OF *m* × *n* GAMES BY LINEAR PROGRAMMING

Following example of (3 × 3) game will make the computational procedure clear.

Example 8. *Solve* (3 × 3) *game by the simplex method of linear programming whose payoff matrix is given below.*

		Player *B*		
		1	2	3
	1	3	– 1	– 3
Player *A*	2	– 3	3	– 1
	3	– 4	– 3	3

[JNTU (B. Tech.) 2004; Meerut (MCA) 2000]

Solution. First apply *minimax (maximin)* criterion to find the minimax ($\overline{v}$) and maximin ($\underline{v}$) value of the game. Thus, the following matrix is obtained (Table 19·8).

Table 19·8

		B			
		1	2	3	Row Minimum.
	1	$\boxed{3}$	– 1	– 3	– 3
A	2	– 3	$\boxed{3}$	– 1	– 3
	3	– 4	– 3	$\boxed{3}$	– 4
Column Maximum		3	3	3	

Maximin Value ($\underline{v}$)

Minimax Value ($\overline{v}$)

Since, maximin value is – 3, it is possible that the value of the game (v) may be negative or zero because

– 3 < v < 3.

Thus, a constant *c* is added to all elements of the matrix which is *at least* equal to the – ve of the maximin value, *i.e.* $c \ge 3$. Let $c = 5$. The matrix is shown in *Table* 1·9. Now, following the reasoning of *Sec.* **19·10,** the player *B*'s linear programming problem is :

Table 19·9

		B		
	1	8	4	2
A	2	2	8	4
	3	1	2	8

$$\text{Maximize } \mathbf{y}_0 = Y_1 + Y_2 + Y_3 \qquad \text{...(19·29)}$$

subject to the constraints :

$$8Y_1 + 4Y_2 + 2Y_3 \le 1 \;,\; 2Y_1 + 8Y_2 + 4Y_3 \le 1 \;,\; 1Y_1 + 2Y_2 + 8Y_3 \le 1 \;,\; Y_1 \ge 0 \;,\; Y_2 \ge 0,\; Y_3 \ge 0 \qquad \text{...(19·30)}$$

Introducing slack variables, the constraint equations become :

$$\left.\begin{array}{l} 8Y_1+4Y_2+2Y_3+Y_4 \qquad\qquad =1 \\ 2Y_1+8Y_2+4Y_3 \qquad +Y_5 \qquad =1 \\ 1Y_1+2Y_2+8Y_3 \qquad\qquad +Y_6 =1 \end{array}\right\} \qquad ...(19{\cdot}31)$$

$$Y_1\,,\,Y_2\,,\,Y_3\,,\,Y_4\,,\,Y_5\,,\,Y_6\ \geq 0.$$

Now the following simplex table is formed.

Table 19·10. Simplex Table

		$c_j \to$	1	1	1	0	0	0	
B	C_B	Y_B	α_1 $\uparrow$	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio (Y_B/α_k)
α_4	0	1	$\leftarrow \boxed{8}$	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$y_0 = C_B Y_B = 0$		$(-1)^*$ $\uparrow$	-1	-1	0 $\downarrow$	0	0	$\leftarrow \Delta_j = C_B\alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	7/2	$-1/4$	1	0	3/14
α_6	0	7/8	0	3/2	$\leftarrow \boxed{31/4}$	1/8	0	1	7/62 ←
	$y_0 = 1/8$		0	$-1/2$	$(-3/4)^*$ $\uparrow$	1/8	0	0 $\downarrow$	$\leftarrow \Delta_j$
α_1	1	3/31	1	14/31	0	4/31	0	$-1/31$	3/14
α_5	0	11/31	0	$\leftarrow \boxed{196/31}$	0	$-6/31$	1	$-14/31$	11/196 ←
α_3	1	7/62	0	6/31	1	$-1/62$	0	4/31	7/12
	$y_0 = 13/62$		0	$(-11/31)^*$ $\uparrow$	0	7/62	0 $\downarrow$	3/31	$\leftarrow \Delta_j$
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	$-3/98$	31/196	$-1/14$	
α_3	1	5/49	0	0	1	$-1/98$	$-3/98$	1/7	
	$y_0 = 45/196$		0	0	0	5/49	11/196	1/14	← all $\Delta_j \geq 0$

Thus, the solution for B's original problem is obtained as :

$$y_1^* = \frac{Y_1}{y_0} = \frac{1/14}{45/196} = \frac{14}{45}\,,\ \ y_2^* = \frac{Y_2}{y_0} = \frac{11/196}{45/196} = \frac{11}{45}.$$

$$y_3^* = \frac{Y_3}{y_0} = \frac{5/49}{45/196} = \frac{20}{45}\,,\ v^* = \frac{1}{y_0} - c = \frac{196}{45} - 5 = -\frac{29}{45}.$$

The optimal strategies for the player A are obtained from the final table of the above problem. This is given by duality rules :

$$x_0 = y_0 = \frac{45}{196}\,,\ X_1 = \Delta_4 = \frac{5}{49}\,,\ X_2 = \Delta_5 = \frac{11}{196}\,,\ X_3 = \Delta_6 = \frac{1}{14}\,.$$

Hence,

$$x_1^* = \frac{X_1}{x_0} = \frac{20}{45}\,,\ x_2^* = \frac{X_2}{x_0} = \frac{11}{45}\,,\ x_3^* = \frac{X_3}{x_0} = \frac{14}{45}\,,\ v^* = \frac{29}{45}.$$

EXAMINATION PROBLEMS

1. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix :
Use linear programming to determine the best strategies for both the players.
[Madurai BSc (Math.) 93; Raj. (M. Phil.) 91]

$$\begin{array}{ccccc} & & & A & \\ & & A_1 & A_2 & A_3 \\ & B_1 & 2 & -2 & 3 \\ B & B_2 & -3 & 5 & -1 \end{array}$$

[Hint. First, make the payoff's positive by adding a constant quantity $c = 4$ (say). The modiefied payoff matrix becomes

$$\begin{array}{cc} & A \\ B & \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix} \end{array}$$

Then, formulate the problem for player B by usual transformation as :

$$\text{Maximize } \mathbf{y_0} = Y_1 + Y_2\text{., subject to : } 6Y_1 + Y_2 \le 1,\ 2Y_1 + 9Y_2 \le 1,\ 7Y_1 + 3Y_2 \le 1, \text{ and } Y_1 \ge 0,\ Y_2 \ge 0$$

Now apply simplex method to find the following solution for B :

$$v = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.\ \ y_1 = \frac{Y_1}{y_0} = \frac{7}{52} \times \frac{13}{3} = \frac{7}{12}.\ \ y_2 = \frac{Y_2}{y_0} = \frac{5}{52} \times \frac{13}{3} = \frac{5}{12}.$$

For player A, read the solution to the dual of above problem

$$v = \frac{1}{\mathbf{x_0}} - 4 = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.\ x_1 = \frac{X_1}{\mathbf{x_0}} = \frac{2}{13} \times \frac{13}{3} = \frac{2}{3}.\ x_2 = \frac{X_2}{\mathbf{x_0}} = \frac{1}{13} \times \frac{13}{3} = \frac{1}{3}.\ x_3 = 0.]$$

[**Ans.** (2/3 , 1/3 , 0) ; (7/12 , 5/12) ; v = 1/3]

2. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :

		Player Q		
		Q_1	Q_2	Q_3
Player P	P_1	9	1	4
	P_2	0	6	3
	P_3	5	2	8

[**Hint.** Payoff's are already non-negative. Formulation of L.P. problem for Q in usual notations is :

$$\text{Max. } \mathbf{y_0} = Y_1 + Y_2 + Y_3, \text{ subject to :}$$

$$9Y_1 + 1Y_2 + 4Y_3 \le 1,\ 0Y_1 + 6Y_2 + 3Y_3 \le 1,$$
$$5Y_1 + 2Y_2 + 8Y_3 \le 1, \text{ and } Y_1, Y_2, Y_3 \ge 0.$$

Its dual is the formulation for player P. Proceeding exactly as in solved example apply simplex method.

[**Ans.** (3/8 , 13/24 , 1/12) ; (7/24 , 5/9 , 11/72) ; v = 91/24]

3. Solve the following games by linear programming :

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

(ii)

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{bmatrix} \end{array}$$

(iii)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

(iv)

$$\begin{array}{cc} & B \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} \end{array}$$

[**Ans.** *(i)* A (17/46 , 20/46 , 9/46); B (7/23 , 6/23 , 10/23) , v = 15/23] *(ii)* (6/11 , 3/11 , 2/11) ; . (5/22 , 8/22 , 9/22) ; v = 6/11] *(iii)* (2/3 , 1/3 , 0), (0, 1/2, 1/2), v = 1]

4. Solve the following 3 × 3 games by linear programming :

(i)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \end{array}$$

[Agra 98, 93, 92]

[**Ans.** A (6/13, 3/13 , 4/13), B (6/13, 4/13, 3/13), v* = – 1/13]

(ii)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

[Meerut 93]

[**Ans.** (0, 0, 1), (4/5 , 1/5 , 0), v = 2]

5. A and B play a game in which each has three coins : a penny, a nickel and a dime. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins *B's coins; If the sum is even, B* wins A's coin. Find the best strategies for each player and the value of game.

$$\left[\textbf{Ans. } A \begin{pmatrix} \text{Penny} & \text{NIckel} & \text{Dime} \\ 1/2 & 1/2 & 0 \end{pmatrix}, B \begin{pmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ 2/3 & 1/3 & 0 \end{pmatrix}, v = 0 \right]$$

6. A and B play a game as follows :

They simultaneously and independently write one of the three numbers 1, 2 and 3. If the sum of the numbers written is even, B pays to A this sum in Rupees. If it is odd, A pays the sum to B in Rupees. Form the payoff matrix of player A and solve the game to find out the value of the game and probabilities of mixed strategies of A and B.

$$\textbf{[Ans. } \begin{pmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{pmatrix}, A(1/4, 1/2, 1/4), B(1/4, 1/2, 1/4), v = 0]$$

7. Convert the following problems into linear programming problem :

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 3 & -2 \\ 2 & 4 & 0 \\ 4 & 5 & 1 \end{bmatrix} \end{array}$$

(ii)

$$\begin{array}{cc} B & \\ A & \begin{bmatrix} 8 & 20 & -3 & 1 \\ 6 & 25 & 4 & 2 \\ 0 & -8 & 12 & 9 \\ 16 & 9 & 21 & 0. \end{bmatrix} \end{array}$$

8. For the following payoff matrix, find the value of the game and the strategies of players A and B by using linear programming :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & -2 \end{bmatrix} \end{array}$$

9. Transform the following matrix games into their corresponding primal and dual linear programming problems. Hence solve them.

(a) $\begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \end{pmatrix}$ **[Delhi BSc (Maths) 91]**

(b) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ **[Delhi B.Sc. (Math.) 91]**

[Ans. (1/2, 1/2), (3/5, 2/5, 0), $v = 1$]

[Ans. (1/2 , 1/2 , 0), (0, 1/2 , 1/2), $v = 0$]

10. Use simplex method to solve the following games :

(a) $\begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

[Ans. (2/3 , 1/2, 0), (0, 1/2 ,1/2), $v = 5$]

[Ans. (0, 0, 1), (4/5, 1/5, 0), $v = 2$]

11. Transform the following matrix game into its corresponding primal and dual linear programming problems :

$$\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

[Delhi M.Sc. (OR) 92]

Solve one of these linear programming problems to obtain the value and the optimal strategies for the two players.

[Ans. Primal. Min. $\mathbf{x_0} = X_1 + X_2$, subject to

$5X_1 + 4X_2 \geq 1,\ 4X_1 + 3X_2 \geq 1,\ 3X_1 + 6X_2 \geq 1,\ X_1 + 5X_2 \geq 1$, and $X_1 \geq 0,\ X_2 \geq 0$

Dual : Max. $\mathbf{y_0} = Y_1 + Y_2 + Y_3$, subject to $5Y_1 + 4Y_2 + 3Y_3 + Y_4 \leq 1$, $4Y_1 + 3Y_2 + 6Y_3 + 5Y_4 \leq 1$,

$Y_i \geq 0,\ i = 1, 2, 3, 4$, and $c = 3$.

12. In a two person game each player simultaneously shows either one or two fingers. If the number of fingers match, player A wins a rupee from player B, otherwise A pays a rupee to B. Show that the payoff matrix for this game is :

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ Solve this game by reducing it to an L.P.P.

[Ans. (1/2, 1/2), (1/2, 1/2), $v = 0$]

13. Two players independently select one of 'mouse', 'cat', 'tiger'. and 'elephant' and simultaneously reveal their choices. It is known that the cat chases the mouse (for score 1), the tiger chases the cat (for score 2), the elephant chases the tiger (for score 3) and the mouse chases the elephant (for a score 4). All other combinations yield a zero score. Formulate the payoff matrix and determine the optimal strategies of the two players.

[Hint. The payoff matrix is skew-symmetric :

$$\begin{array}{l|cccc|} & \text{Mouse} & \text{Cat} & \text{Tiger} & \text{Elephant} \\ \text{Mouse} & 0 & -1 & 0 & 4 \\ \text{Cat} & 1 & 0 & -2 & 0 \\ \text{Tiger} & 0 & 2 & 0 & -3 \\ \text{Elephant} & -4 & 0 & 3 & 0 \end{array}$$

14. Solve by using L.P. process, whose pay-off matrix is

$$\begin{array}{c} \\ A \end{array} \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$$

[Meerut (M.A.) 97]

15. For the following pay-off matrix, find the value of the game and the strategies of players *A* and *B* by using linear programming :

$$\begin{array}{cc|ccc|} & & & \textbf{Player B} & \\ & & 1 & 2 & 3 \\ \textbf{Player } A & 1 & 3 & -1 & 4 \\ & 2 & 6 & 7 & -2 \end{array}$$

[Delhi (M.B.A.) 96]

[Ans. The solution to the problem, therefore, is : $S_A = (9/14, 5/14)$, $S_B = (0, 3/7, 4/7)$, value of game = 13/7.

19·13. TWO-BY-TWO (2 × 2) GAMES WITHOUT SADDLE POINT

There are several methods for determining the optimal strategies and the value of the game. But, in most of the situations, the matrix game can be reduced to a 2 × 2 game (to be discussed later in *Secs.* 19·14 & 19·15). It is therefore worth- while to determine the solution of 2 × 2 game in the following theorem.

Theorem 19·3. *Show that for any zero-sum two-person game where optimal strategies are not pure strategies (i.e. there is no saddle point) and for which the player A's payoff matrix is*

$$\begin{array}{cc|cc|} & & \text{B} & \\ & & y_1 & y_2 \\ \text{A} & x_1 & v_{11} & v_{12} \\ & x_2 & v_{21} & v_{22} \end{array}$$

and optimal strategies (x_1 , x_2) *and* (y_1 , y_2) *are determined by*

$$\frac{x_1}{x_2}=\frac{v_{22}-v_{21}}{v_{11}-v_{12}},\quad \frac{y_1}{y_2}=\frac{v_{22}-v_{12}}{v_{11}-v_{21}}.$$

and the value (v) *of the game to the player A is given by*

$$v=\frac{v_{11}v_{22}-v_{12}v_{21}}{v_{11}+v_{22}-(v_{12}+v_{21})}.$$

[Meerut 2002; Rohilkhand 92]

Proof. Let a mixed strategy for player A be given by (x_1 , x_2) where $x_1 + x_2 = 1$. Thus if player B moves his first strategy, the net expected gain of A will be $E_1 (x) = v_{11}x_1 + v_{21}x_2$;
and if B moves his second strategy, the net expected gain of A will be $E_2 (x) = v_{12}x_1 + v_{22}x_2$.

But, player A wants to maximize his minimum expected gain. So the value of the game (v) must be minimum of $E_1(x)$ and $E_2(x)$, *i.e.* $E_1(x) \geq v$, $E_2(x) \geq v$.

Thus for the player A, we have to find $x_1 \geq 0, x_2 \geq 0$, and v to satisfy the following three relationships (as obtained in ***Sec.19·10***) :

$$v_{11}x_1 + v_{21}x_2 \geq v , \qquad \ldots(19{\cdot}33)$$

$$v_{12}x_1 + v_{22}x_2 \geq v , \qquad \ldots(19{\cdot}34)$$

$$x_1 + x_2 = 1. \qquad \ldots(19{\cdot}35)$$

For optimum strategies, inequalities (19·33) and (19·34) become strict equations, *i.e.*

$$v_{11}x_1 + v_{21}x_2 = v, \qquad \ldots(19{\cdot}36)$$

$$v_{12}x_1 + v_{22}x_2 = v . \qquad \ldots(19{\cdot}37)$$

Subtracting equation (19·37) from the equation (19·36), we get

$$(v_{11} - v_{12})\, x_1 + (v_{21} - v_{22})\, x_2 = 0 . \qquad \ldots(19{\cdot}38)$$

which gives

$$\frac{x_1}{x_2}=\frac{v_{22}-v_{21}}{v_{11}-v_{12}}. \qquad \ldots(19{\cdot}39)$$

Hence, we evaluate x_1 and x_2 separately by using the equation (19·35),

$$x_1=\frac{v_{22}-v_{21}}{v_{11}+v_{22}-(v_{12}+v_{21})} \qquad \ldots(19{\cdot}40)$$

$$x_2=1-x_1=\frac{v_{11}-v_{12}}{(v_{11}+v_{22})-(v_{12}+v_{21})}. \qquad \ldots(19{\cdot}41)$$

The value of the game can be obtained by substituting the values of x_1 and x_2 in either of the equations (19·36) and (19·37) to obtain

$$v=\frac{v_{11}\,(v_{22}-v_{21})}{v_{11}+v_{22}-(v_{12}+v_{21})}+\frac{v_{21}\,(v_{11}-v_{12})}{v_{11}+v_{22}-(v_{12}+v_{21})}\ \text{or}\ v=\frac{v_{11}v_{22}-v_{21}v_{12}}{v_{11}+v_{22}-(v_{12}+v_{21})}. \qquad \ldots(19{\cdot}42)$$

In the same manner for the player B, find $y_1 \geq 0$, $y_2 \geq 0$, and v to satisfy the following three relations :

$$v_{11}y_1 + v_{12}y_2 \leq v , \qquad \ldots(19{\cdot}43)$$

$$v_{21}y_1 + v_{22}y_2 \leq v , \qquad \ldots(19{\cdot}44)$$

$$y_1 + y_2 = 1. \qquad \ldots(19{\cdot}45)$$

Here it should be remembered that the player B wants to minimize his maximum loss.

Again for optimum strategies of player B, consider the inequalities (19·43) and (19·44) as strict equations and obtain

$$\frac{y_1}{y_2}=\frac{v_{22}-v_{12}}{v_{11}-v_{21}}. \qquad \ldots(19{\cdot}46)$$

Using the equation (19·45)

$$y_1=\frac{v_{22}-v_{12}}{v_{11}+v_{22}-(v_{21}+v_{12})} \qquad \ldots(19{\cdot}47)$$

$$y_2=1-y_1=\frac{v_{11}-v_{21}}{v_{11}+v_{22}-(v_{21}+v_{12})}. \qquad \ldots(19{\cdot}48)$$